

ENCYCLOPEDIA OF TYPES OF ALGEBRAS 2010

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ABSTRACT. This is a cornucopia of types of algebras with some of their properties from the operadic point of view.

INTRODUCTION

The following is a list of some types of algebras together with their properties under an operadic and homological point of view. We keep the information to one page per type and we provide one reference as Ariadne's thread. More references are listed by the end of the paper. We work over a fixed field \mathbb{K} though in many instances everything makes sense and holds over a commutative ground ring. The category of vector spaces over \mathbb{K} is denoted by **Vect**. All tensor products are over \mathbb{K} unless otherwise stated.

The items of a standard page (which is to be found at the end of this introduction) are as follows.

Sometimes a given type appears under different names in the literature. The choice made in **Name** is, most of the time, the most common one (up to a few exceptions). The other possibilities appear under the item **Alternative**.

The presentation given in **Definit.** is the most common one (with one exception). When others are used in the literature they are given in **Alternative**. The item **oper.** gives the generating operations. The item **sym.** gives their symmetry properties, if any. The item **rel.** gives the relation(s). They are supposed to hold for any value of the variables x, y, z, \dots . If, in the presentation, only binary operations appear, then the type is said to be *binary*. Analogously, there are *ternary*, *k-ary*, *multi-ary* types.

If, in the presentation, the relations involve only the composition of two operations at a time (hence 3 variables in the binary case, 5 variables in the ternary case), then the type is said to be *quadratic*.

For a given type of algebras \mathcal{P} the category of \mathcal{P} -algebras is denoted by **\mathcal{P} -alg**. For each type there is defined a notion of *free algebra*. By definition the free algebra of type \mathcal{P} over the vector space V is an algebra denoted by $\mathcal{P}(V)$ satisfying the following universal condition:

for any algebra A of type \mathcal{P} and any linear map $\phi : V \rightarrow A$ there is a unique \mathcal{P} -algebra morphism $\tilde{\phi} : \mathcal{P}(V) \rightarrow A$ which lifts ϕ . In other words the

forgetful functor

$$\mathcal{P}\text{-alg} \longrightarrow \mathbf{Vect}$$

admits a left adjoint

$$\mathcal{P} : \mathbf{Vect} \longrightarrow \mathcal{P}\text{-alg}.$$

In all the cases mentioned here the relations involved in the presentation of the given type are (or can be made) multilinear. Hence the functor $\mathcal{P}(V)$ is of the form (at least in characteristic zero),

$$\mathcal{P}(V) = \bigoplus_{n \geq 1} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n},$$

where $\mathcal{P}(n)$ is some \mathbb{S}_n -module. The \mathbb{S}_n -module $\mathcal{P}(n)$ is called the space of n -ary operations since for any algebra A there is a map

$$\mathcal{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \rightarrow A.$$

The functor $\mathcal{P} : \mathbf{Vect} \rightarrow \mathbf{Vect}$ inherits a monoid structure from the properties of the free algebra. Hence there exist transformations of functors $\iota : \text{Id} \rightarrow \mathcal{P}$ and $\gamma : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$ such that γ is associative and unital. The monoid $(\mathcal{P}, \gamma, \iota)$ is called a *symmetric operad*.

The symmetric operad \mathcal{P} can also be described as a family of \mathbb{S}_n -modules $\mathcal{P}(n)$ together with maps

$$\gamma(i_1, \dots, i_k) : \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \dots \otimes \mathcal{P}(i_k) \longrightarrow \mathcal{P}(i_1 + \dots + i_k)$$

satisfying some compatibility with the action of the symmetric group and satisfying the associativity property.

If \mathbb{S}_n is acting freely on $\mathcal{P}(n)$, then $\mathcal{P}(n) = \mathcal{P}_n \otimes \mathbb{K}[\mathbb{S}_n]$ where \mathcal{P}_n is some vector space, and $\mathbb{K}[\mathbb{S}_n]$ is the regular representation. If, moreover, the maps $\gamma(i_1, \dots, i_n)$ are induced by maps

$$\gamma_{i_1, \dots, i_k} : \mathcal{P}_k \otimes \mathcal{P}_{i_1} \otimes \dots \otimes \mathcal{P}_{i_k} \longrightarrow \mathcal{P}_{i_1 + \dots + i_k},$$

then the operad \mathcal{P} comes from a *nonsymmetric operad* (abbreviated ns operad), still denoted \mathcal{P} .

For more terminology and details about algebraic operads we refer to [LV11].

The generating series of the operad \mathcal{P} is defined as

$$f^{\mathcal{P}}(t) := \sum_{n \geq 1} \frac{\dim \mathcal{P}(n)}{n!} t^n,$$

in the binary case. When dealing with a nonsymmetric operad it becomes

$$f^{\mathcal{P}}(t) := \sum_{n \geq 1} \dim \mathcal{P}_n t^n.$$

The Koszul duality theory of associative algebras has been extended to binary quadratic operads by Ginzburg and Kapranov, cf. [GK94], then to quadratic operads by Fresse, cf. [Fre06]. We give a conceptual treatment of this theory, together with applications in [LV11]. So, to any quadratic operad \mathcal{P} , there is associated a quadratic **dual operad** denoted $\mathcal{P}^!$. It is

often a challenge to find a presentation of $\mathcal{P}^!$ out of a presentation of \mathcal{P} . One of the main results of the Koszul duality theory of operads is to show the existence of a natural differential map on the composite $\mathcal{P}^{!*} \circ \mathcal{P}$ given rise to the *Koszul complex*. If it is acyclic, then \mathcal{P} is said to be *Koszul*. One can show that, if \mathcal{P} is Koszul, then so is $\mathcal{P}^!$. In this case the generating series are inverse to each other for composition, up to sign, that is:

$$f^{\mathcal{P}^!}(-f^{\mathcal{P}}(-t)) = t.$$

In the k -ary case we introduce the skew-generating series

$$g^{\mathcal{P}}(t) := \sum_{n \geq 1} (-1)^k \frac{\dim \mathcal{P}((k-1)n+1)}{n!} t^{((k-1)n+1)}.$$

If the operad \mathcal{P} is Koszul, then (cf. [Val07])

$$f^{\mathcal{P}^!}(-g^{\mathcal{P}}(-t)) = t.$$

The items **Free alg.**, **rep.** $\mathcal{P}(n)$ or \mathcal{P}_n , $\dim \mathcal{P}(n)$ or $\dim \mathcal{P}_n$, and **Gen. series** speak for themselves.

Koszulity of an operad implies the existence of a small chain complex to compute the (co)homology of a \mathcal{P} -algebra. When possible, the information on it is given in the item **Chain-cplx**. Moreover it permits us to construct the notion of \mathcal{P} -algebra up to homotopy, whose associated operad, which is a differential graded operad, is denoted \mathcal{P}_∞ .

The item **Properties** lists the main features of the operad. *Set-theoretic* means that there is a set operad \mathcal{P}_{Set} (monoid in the category of \mathbb{S} -Sets) such that $\mathcal{P} = \mathbb{K}[\mathcal{P}_{Set}]$. Usually this property can be read on the presentation of the operad: no algebraic sums.

In the item **Relationship** we list some of the ways to obtain this operad under some natural constructions like tensor product (Hadamard product) or Manin products (white \circ or black \bullet), denoted \square and \blacksquare in the nonsymmetric framework, cf.[LV11]) for instance. We also list some of the most common functors to other types of algebras. Keep in mind that a functor $\mathcal{P} \rightarrow \mathcal{Q}$ induces a functor $\mathcal{Q}\text{-alg} \rightarrow \mathcal{P}\text{-alg}$ on the categories of algebras.

Though we describe only algebras without unit, for some types there is a possibility of introducing an element 1 which is either a unit or a partial unit for some of the operations, see the discussion in [Lod04]. We indicate it in the item **Unit**.

For binary operads the *opposite type* consists in defining new operations by $x \cdot y = yx$, etc. If the new type is isomorphic to the former one, then the operad is said to be *self-opposite*. When it is not the case, we mention whether the given type is called *right* or *left* in the item **Comment**.

In some cases the structure can be “integrated”. For instance Lie algebras are integrated into Lie groups (Lie third problem). If so, we indicate it in the item **Comment**.

In the item **Ref.** we indicate a reference where information on the operad and/or on the (co)homology theory can be obtained. It is not necessarily the

first paper in which this type of algebras first appeared. For the three grades As , Com , Lie , the classical books Cartan-Eilenberg “Homological Algebra” and MacLane “Homology” are standard references.

Here is the list of the types included so far (with page number and letter K indicating that they are Koszul dual to each other):

<i>sample</i>	6	<i>As</i>	7	self-dual
<i>Com</i>	8	<i>Lie</i>	9	K
<i>Pois</i>	10	none	11	self-dual
<i>Leib</i>	12	<i>Zinb</i>	13	K
<i>Dend</i>	14	<i>Dias</i>	15	K
<i>PreLie</i>	16	<i>Perm</i>	17	K
<i>Dipt</i>	18	<i>Dipt</i> [!]	19	K
<i>2as</i>	20	<i>2as</i> [!]	21	K
<i>Tridend</i>	22	<i>Trias</i>	23	K
<i>PostLie</i>	24	<i>ComTrias</i>	25	K
<i>CTD</i>	26	<i>CTD</i> [!]	27	K
<i>Gerst</i>	28	<i>BV</i>	29	
<i>Mag</i>	30	<i>Nil</i> ₂	31	K
<i>ComMag</i>	32	<i>ComMag</i> [!]	33	K
<i>Quadri</i>	34	<i>Quadri</i> [!]	35	K
<i>Dup</i>	36	<i>Dup</i> [!]	37	K
<i>As</i> ⁽²⁾	38	<i>As</i> ⁽²⁾	39	
<i>L-dend</i>	40	<i>Lie-adm</i>	41	
<i>PreLiePerm</i>	42	<i>Altern</i>	43	
<i>Param1rel</i>	44	<i>MagFine</i>	45	
<i>GenMag</i>	46	<i>NAP</i>	47	
<i>Moufang</i>	48	<i>Malcev</i>	49	
<i>Novikov</i>	50	<i>DoubleLie</i>	51	
<i>DiPreLie</i>	52	<i>Akivis</i>	53	
<i>Sabinin</i>	54	<i>Jordan triples</i>	55	
<i>t-As</i> ⁽³⁾	56	<i>p-As</i> ⁽³⁾	57	
<i>LTS</i>	58	<i>Lie-Yamaguti</i>	59	
<i>Interchange</i>	60	<i>HyperCom</i>	61	
<i>A</i> _∞	62	<i>C</i> _∞	63	
<i>L</i> _∞	64	<i>Dend</i> _∞	65	
<i>P</i> _∞	66	<i>Brace</i>	67	
<i>MB</i>	68	<i>2Pois</i>	69	
<i>Ξ</i> [±]	70	your own	71	

Notation

We use the notation \mathbb{S}_n for the symmetric group. Trees are very much in use in the description of operads. We use the following notation:

– PBT_n is the set of planar binary rooted trees with $n-1$ internal vertices (and hence n leaves). The number of elements of PBT_n is the Catalan number $c_n = \frac{1}{n+1} \binom{2n}{n}$.

– PT_n is the set of planar rooted trees with n leaves, whose vertices have valency greater than 1 + 2 (one root, at least 2 inputs). So we have $PBT_n \subset PT_n$. The number of elements of PT_n is the super Catalan number, also called Schröder number, denoted C_n .

A planar binary rooted tree t is completely determined by its right part t^r and its left part t^l . More precisely t is the grafting of t^l and t^r :

$$t = t^l \vee t^r .$$

Comments

Many thanks to Walter Moreira for setting up a software which computes the first dimensions of the operad from its presentation.

We remind the reader that we can replace the symmetric monoidal category **Vect** by many other symmetric monoidal categories. So there are notions of graded algebras, differential graded algebras, twisted algebras, and so forth. In the graded cases the Koszul sign rule is in order. Observe that there are also operads where the operations may have different degree (operad encoding Gerstenhaber algebras for instance).

We end this paper with a tableau of integer sequences appearing in this document.

This list of types of algebras is not as encyclopedic as the title suggests. We put only the types which are defined by a finite number of generating operations and whose relations are multilinear. Moreover we only put those which have been used some way or another. You will not find the “restricted types” (like divided power algebras), nor bialgebras.

Please report any error or comment to: `gw.zinbiel@free.fr`

We plan to update this encyclopedia every now and then.

1. TYPE OF ALGEBRAS

Name **Most common terminology**

Notation my favorite notation for the operad (generic notation: \mathcal{P})

Def. oper. list of the generating operations
 sym. their symmetry if any
 rel. the relation(s)

Free alg. the free algebra as a functor in V

rep. $\mathcal{P}(n)$ the \mathbb{S}_n -representation $\mathcal{P}(n)$ and/or the space \mathcal{P}_n if nonsymmetric

dim $\mathcal{P}(n)$ the series (if close formula available), the list of the 7 first numbers beginning at $n = 1$

Gen. series close formula for $f^{\mathcal{P}}(t) = \sum_{n \geq 1} \frac{\dim \mathcal{P}(n)}{n!} t^n$ when available

Dual operad the Koszul dual operad

Chain-cplx Explication of the chain complex, if not too complicated

Properties among: nonsymmetric, binary, quadratic, set-theoretic, ternary, multi-ary, cubic, Koszul.

Alternative alternative terminology, and/or notation, and/or presentation.

Relationship some of the relationships with other operads, either under some construction like symmetrizing, Hadamard product, Manin products, or under the existence of functors

Unit whether one can assume the existence of a unit (or partial unit)

Comment whatever needs to be said which does not fit into the other items

Ref. a reference, usually dealing with the homology of the \mathcal{P} -algebras (not necessarily containing all the results of this page)

Name	Associative algebra
Notation	As (as nonsymmetric operad) Ass (as symmetric operad)
Def. oper.	xy , operadically μ
sym.	
rel.	$(xy)z = x(yz)$, operadically $\mu \circ_1 \mu = \mu \circ_2 \mu$ (associativity)
Free alg.	$As(V) = \overline{T}(V) = \bigoplus_{n \geq 1} V^{\otimes n}$ tensor algebra (noncommutative polynomials) $(x_1 \dots x_p)(x_{p+1} \dots x_{p+q}) = x_1 \dots x_{p+q}$ (concatenation)
rep. $\mathcal{P}(n)$	$As(n) = \mathbb{K}[\mathbb{S}_n]$ (regular representation), $As_n = \mathbb{K}$
$\dim \mathcal{P}(n)$	$1, 2, 6, 24, 120, 720, 5040, \dots, n!, \dots$
Gen. series	$f^{As}(t) = \frac{t}{1-t}$
Dual operad	$As^! = As$
Chain-cplx	non-unital Hochschild complex, $C_n^{As}(A) = A^{\otimes n}$ $b'(a_1, \dots, a_n) = \sum_{i=1}^{n-1} (-1)^{i-1} (a_1, \dots, a_i a_{i+1}, \dots, a_n)$ important variation: cyclic homology
Properties	binary, quadratic, ns, set-theoretic, Koszul, self-dual.
Alternative	Associative algebra often simply called <i>algebra</i> . Can be presented with commutative operation $x \cdot y := xy + yx$ and anti-symmetric operation $[x, y] = xy - yx$ satisfying $\begin{cases} [x \cdot y, z] = x \cdot [y, z] + [x, z] \cdot y, \\ (x \cdot y) \cdot z - x \cdot (y \cdot z) = [x, [y, z]] \end{cases}$ (Livernet and Loday, unpublished)
Relationship	$Ass\text{-alg} \rightarrow Lie\text{-alg}$, $[x, y] = xy - yx$, $Com\text{-alg} \rightarrow Ass\text{-alg}$ (inclusion), and many others
Unit	$1x = x = x1$
Comment	one the three graces

Name	Commutative algebra
Notation	Com
Def. oper.	xy
sym.	$xy = yx$ (commutativity)
rel.	$(xy)z = x(yz)$ (associativity)
Free alg.	$Com(V) = \overline{S}(V)$ (polynomials) if $V = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$, then $\mathbb{K}1 \oplus Com(V) = \mathbb{K}[x_1, \dots, x_n]$
rep. $\mathcal{P}(n)$	$Com(n) = \mathbb{K}$ (trivial representation)
$\dim \mathcal{P}(n)$	$1, 1, 1, 1, 1, 1, 1, \dots, 1, \dots$
Gen. series	$f^{Com}(t) = \exp(t) - 1$
Dual operad	$Com^! = Lie$
Chain-cplx	Harrison complex in char. 0, André-Quillen cplx in general
Properties	binary, quadratic, set-theoretic, Koszul.
Alternative	sometimes called associative and commutative algebra other notation $Comm$
Relationship	$Com\text{-alg} \rightarrow As\text{-alg}$, $Zinb\text{-alg} \rightarrow Com\text{-alg}$
Unit	$1x = x = x1$
Comment	one of the three graces

Name	Lie algebra
Notation	Lie
Def. oper.	$[x, y]$ (bracket)
sym.	$[x, y] = -[y, x]$ (anti-symmetry)
rel.	$[[x, y], z] = [x, [y, z]] + [[x, z], y]$ (Leibniz relation)
Free alg.	$Lie(V)$ = subspace of the tensor algebra $T(V)$ generated by V under the bracket
rep. $\mathcal{P}(n)$	$Lie(n) = \text{Ind}_{C_n}^{\mathbb{S}_n}(\sqrt[n]{1})$
$\dim \mathcal{P}(n)$	$1, 1, 2, 6, 24, 120, 720, \dots, (n-1)!, \dots$
Gen. series	$f^{Lie}(t) = -\log(1-t)$
Dual operad	$Lie^! = Com$
Chain-cplx	Chevalley-Eilenberg complex $C_n^{Lie}(\mathfrak{g}) = \Lambda^n \mathfrak{g}$ $d(x_1 \wedge \dots \wedge x_n) = \sum_{1 \leq i < j \leq n} (-1)^j (x_1 \wedge \dots \wedge [x_i, x_j] \wedge \dots \wedge \widehat{x_j} \wedge \dots \wedge x_n)$
Properties	binary, quadratic, Koszul.
Alternative	The relation is more commonly written as the Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
Relationship	$As\text{-alg} \rightarrow Lie\text{-alg},$ $Lie\text{-alg} \rightarrow Leib\text{-alg},$ $PreLie\text{-alg} \rightarrow Lie\text{-alg}$
Unit	no
Comment	one of the three graces. Named after Sophus Lie. Integration: Lie groups.

Name	Poisson algebra
Notation	$Pois$
Def. oper.	$xy, \{x, y\}$
sym.	$xy = yx, \{x, y\} = -\{y, x\}$
rel.	$\begin{cases} \{\{x, y\}, z\} = \{x, \{y, z\}\} + \{\{x, z\}, y\}, \\ \{xy, z\} = x\{y, z\} + \{x, z\}y, \\ (xy)z - x(yz) = 0. \end{cases}$
Free alg.	$Pois(V) \cong \overline{T}(V)$ (tensor module, iso as Schur functors)
rep. $\mathcal{P}(n)$	$Pois(n) \cong \mathbb{K}[\mathbb{S}_n]$ (regular representation)
$\dim \mathcal{P}(n)$	$1, 2!, 3!, 4!, 5!, 6!, 7!, \dots, n!, \dots$
Gen. series	$f^{Pois}(t) = \frac{t}{1-t}$
Dual operad	$Pois^! = Pois$
Chain-cplx	Isomorphic to the total complex of a certain bicomplex constructed from the action of the Eulerian idempotents
Properties	binary, quadratic, quasi-regular, Koszul, self-dual.
Alternative	Can be presented with one operation $x * y$ satisfying the relation $(x * y) * z = x * (y * z) + \frac{1}{3} \left(+ x * (z * y) - z * (x * y) - y * (x * z) + y * (z * x) \right)$
Relationship	$Pois\text{-alg} \rightleftharpoons Lie\text{-alg}, \quad Pois\text{-alg} \rightleftharpoons Com\text{-alg},$
Unit	$1x = x = x1, [1, x] = 0 = [x, 1]$
Comment	Named after Siméon Poisson.
Ref.	[Fre06] B. Fresse, <i>Théorie des opérades de Koszul et homologie des algèbres de Poisson</i> , Ann.Math. Blaise Pascal 13 (2006), 237–312.

This page is inserted so that, in the following part, an operad and its dual appear on page $2n$ and $2n + 1$ respectively. Since *Pois* is self-dual there is no point to write a page about its dual.

Let us take the opportunity to mention that if A is a \mathcal{P} -algebra and B is a $\mathcal{P}^!$ -algebra, then the tensor product $A \otimes B$ inherits naturally a structure of Lie algebra. If \mathcal{P} is nonsymmetric, then so is $\mathcal{P}^!$, and $A \otimes B$ is in fact an associative algebra.

In some cases (like the Leibniz case for instance), $A \otimes B$ is a pre-Lie algebra.

Name	Leibniz algebra
Notation	$Leib$
Def. oper.	$[x, y]$
sym.	
rel.	$[[x, y], z] = [x, [y, z]] + [[x, z], y]$ (Leibniz relation)
Free alg.	$Leib(V) \cong \overline{T}(V)$ (reduced tensor module, iso as Schur functors)
rep. $\mathcal{P}(n)$	$Leib(n) = \mathbb{K}[\mathbb{S}_n]$ (regular representation)
$\dim \mathcal{P}(n)$	$1, 2!, 3!, 4!, 5!, 6!, 7!, \dots, n!, \dots$
Gen. series	$f^{Leib}(t) = \frac{t}{1-t}$
Dual operad	$Leib^! = Zinb$
Chain-cplx	$C_n^{Leib}(\mathfrak{g}) = \mathfrak{g}^{\otimes n}$ $d(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n} (-1)^j (x_1, \dots, [x_i, x_j], \dots, \widehat{x_j}, \dots, x_n)$
Properties	binary, quadratic, quasi-regular, Koszul.
Alternative	Sometimes improperly called Loday algebra.
Relationship	$Leib = Perm \circ Lie$ (Manin white product), see [Val08] $Lie\text{-alg} \rightarrow Leib\text{-alg}$, $Dias\text{-alg} \rightarrow Leib\text{-alg}$, $Dend\text{-alg} \rightarrow Leib\text{-alg}$
Unit	no
Comment	Named after G.W. Leibniz. This is the <i>left</i> Leibniz algebra. The opposite type is called <i>right</i> Leibniz algebra. Integration : “coquecigrues” ! see for instance [Cov10]
Ref.	[Lod93] J.-L. Loday, <i>Une version non commutative des algèbres de Lie: les algèbres de Leibniz</i> . Enseign. Math. (2) 39 (1993), no. 3-4, 269–293.

Name	Zinbiel algebra
Notation	$Zinb$
Def. oper.	$x \cdot y$
sym.	
rel.	$(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y)$ (Zinbiel relation)
Free alg.	$Zinb(V) = \overline{T}(V)$, \cdot = halfshuffle $x_1 \dots x_p \cdot x_{p+1} \dots x_{p+q} = x_1 \text{sh}_{p-1,q}(x_2 \dots x_p, x_{p+1} \dots x_{p+q})$
rep. $\mathcal{P}(n)$	$Zinb(n) \cong \mathbb{K}[\mathbb{S}_n]$ (regular representation)
dim $\mathcal{P}(n)$	1, 2!, 3!, 4!, 5!, 6!, 7!, ..., n!, ...
Gen. series	$f^{Zinb}(t) = \frac{t}{1-t}$
Dual operad	$Zinb^! = Leib$
Chain-cplx	
Properties	binary, quadratic, quasi-regular, Koszul.
Alternative	$Zinb = ComDend$ (commutative dendriform algebra) Previously called <i>dual Leibniz algebra</i> .
Relationship	$Zinb = PreLie \bullet Com$, see [Val08], $Zinb = ComDend$ $Zinb\text{-alg} \rightarrow Com\text{-alg}$, $xy = x \cdot y + y \cdot x$, $Zinb\text{-alg} \rightarrow Dend\text{-alg}$, $x \prec y = x \cdot y = y \succ x$
Unit	$1 \cdot x = 0$, $x \cdot 1 = x$
Comment	symmetrization of the dot product gives, not only a commutative alg., but in fact a <i>divided power algebra</i> . Named after G.W. Zinbiel. This is right Zinbiel algebra.
Ref.	[Lod95] J.-L. Loday, <i>Cup-product for Leibniz cohomology and dual Leibniz algebras</i> . Math. Scand. 77 (1995), no. 2, 189–196.

Name	Dendriform algebra
Notation	$Dend$
Def. oper. sym.	$x \prec y, x \succ y$ (left and right operation)
rel.	$\begin{cases} (x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z), \\ (x \succ y) \prec z = x \succ (y \prec z), \\ (x \prec y) \succ z + (x \succ y) \succ z = x \succ (y \succ z). \end{cases}$
Free alg.	$Dend(V) = \bigoplus_{n \geq 1} \mathbb{K}[PBT_{n+1}] \otimes V^{\otimes n}$, for pb trees s and t : $s \prec t := s^l \vee (s^r * t)$, and $s \succ t := (s * t^l) \vee t^r$ where $x * y := x \prec y + x \succ y$.
rep. $\mathcal{P}(n)$	$Dend(n) = \mathbb{K}[PBT_{n+1}] \otimes \mathbb{K}[\mathbb{S}_n]$, so $Dend_n = \mathbb{K}[PBT_{n+1}]$
$\dim \mathcal{P}(n)$	$1, 2 \times 2!, 5 \times 3!, 14 \times 4!, 42 \times 5!, 132 \times 6!, 429 \times 7!, \dots, c_n \times n!, \dots$ where $c_n = \frac{1}{n+1} \binom{2n}{n}$ is the Catalan number
Gen. series	$f^{Dend}(t) = \frac{1-2t-\sqrt{1-4t}}{2t} = y, \quad y^2 - (1-2t)y + t = 0$
Dual operad	$Dend^! = Dias$
Chain-cplx	Isomorphic to the total complex of a certain explicit bicomplex
Properties	binary, quadratic, ns, Koszul.
Alternative	Handy to introduce $x * y := x \prec y + x \succ y$ which is associative.
Relationship	$Dend = PreLie \bullet As$, see [Val08] $Dend\text{-alg} \rightarrow As\text{-alg}, \quad x * y := x \prec y + y \succ x,$ $Zinb\text{-alg} \rightarrow Dend\text{-alg}, \quad x \prec y := x \cdot y =: y \succ x$ $Dend\text{-alg} \rightarrow PreLie\text{-alg}, \quad x \circ y := x \prec y - y \succ x$ $Dend\text{-alg} \rightarrow Brace\text{-alg}, \quad \text{see Ronco [Ron00]}$
Unit	$1 \prec x = 0, x \prec 1 = x, \quad 1 \succ x = x, x \succ 1 = 0.$
Comment	dendro = tree in greek. There exist many variations.
Ref.	[Lod01] J.-L. Loday, <i>Dialgebras</i> , Springer Lecture Notes in Math. 1763 (2001), 7-66.

Name	Diassociative algebra
Notation	$Dias$
Def. oper. sym.	$x \dashv y, x \vdash y$ (left and right operation)
rel.	$\left\{ \begin{array}{l} (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (x \vdash y) \vdash z = x \vdash (y \vdash z). \end{array} \right.$
Free alg.	$Dias(V) = \bigoplus_{n \geq 1} \underbrace{(V^{\otimes n} \oplus \dots \oplus V^{\otimes n})}_{n \text{ copies}}$ <p>noncommutative polynomials with one variable marked</p>
rep. $\mathcal{P}(n)$	$Dias(n) = \mathbb{K}^n \otimes \mathbb{K}[\mathbb{S}_n], \quad Dias_n = \mathbb{K}^n$
dim $\mathcal{P}(n)$	$1, 2 \times 2!, 3 \times 3!, 4 \times 4!, 5 \times 5!, 6 \times 6!, 7 \times 7!, \dots, n \times n!, \dots,$
Gen. series	$f^{Dias}(t) = \frac{t}{(1-t)^2}$
Dual operad	$Dias^! = Dend$
Chain-cplx	see ref.
Properties	binary, quadratic, ns, set-theoretic, Koszul.
Alternative	Also called associative dialgebras, or for short, dialgebras.
Relationship	$Dias = Perm \circ As = Perm \underset{H}{\otimes} As$ $As\text{-alg} \rightarrow Dias\text{-alg}, \quad x \dashv y := xy =: x \vdash y$ $Dias\text{-alg} \rightarrow Leib\text{-alg}, \quad [x, y] := x \dashv y - x \vdash y$
Unit	Bar-unit: $x \dashv 1 = x = 1 \vdash x, 1 \dashv x = 0 = x \vdash 1$
Comment	
Ref.	[Lod01] J.-L. Loday, <i>Dialgebras</i> , Springer Lecture Notes in Math. 1763 (2001), 7-66.

Name	Pre-Lie algebra
Notation	$PreLie$
Def. oper.	$x \circ y$
sym.	
rel.	$(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y)$
Free alg.	$PreLie(V) = \{\text{rooted trees labeled by elements of } V\}$
rep. $\mathcal{P}(n)$	$PreLie(n) = \mathbb{K}[\{\text{rooted trees, vertices labeled by } 1, \dots, n\}]$
dim $\mathcal{P}(n)$	$1, 2, 9, 64, 625, 1296, 117649, \dots, n^{n-1}, \dots$
Gen. series	$f^{PreLie}(t) = y$ which satisfies $y = t \exp(y)$
Dual operad	$PreLie^! = Perm$
Chain-cplx	see ref.
Properties	binary, quadratic, Koszul.
Alternative	The relation is $as(x, y, z) = as(x, z, y)$.
Relationship	$PreLie\text{-alg} \rightarrow Lie\text{-alg}, \quad [x, y] = x \circ y - y \circ x$ $Dend\text{-alg} \rightarrow PreLie\text{-alg}, \quad x \circ y := x \prec y - y \succ x$ $Brace\text{-alg} \rightarrow PreLie\text{-alg}, \quad \text{forgetful functor}$
Unit	$1 \circ x = x = x \circ 1$
Comment	This is <i>right</i> pre-Lie algebra, also called <i>right-symmetric algebra</i> , or <i>Vinberg algebra</i> . The opposite type is left symmetric. First appeared in [Ger63, Vin63].
Ref.	[CL01] F. Chapoton, M. Livernet, <i>Pre-Lie algebras and the rooted trees operad</i> . Internat. Math. Res. Notices 2001, no. 8, 395–408.

Name	Perm algebra
Notation	$Perm$
Def. oper.	xy
sym.	
rel.	$(xy)z = x(yz) = x(zy)$
Free alg.	$Perm(V) = V \otimes S(V)$
rep. $\mathcal{P}(n)$	$Perm(n) = \mathbb{K}^n$
$\dim \mathcal{P}(n)$	$1, 2, 3, 4, 5, 6, 7, \dots, n, \dots$
Gen. series	$f^{Perm}(t) = t \exp(t)$
Dual operad	$Perm^! = PreLie$
Chain-cplx	
Properties	binary, quadratic, set-theoretic, Koszul.
Alternative	
Relationship	$Perm = ComDias$ $Com\text{-}alg \rightarrow Perm\text{-}alg,$ $NAP\text{-}alg \rightarrow Perm\text{-}alg,$ $Perm\text{-}alg \rightarrow Dias\text{-}alg$
Unit	no, unless it is a commutative algebra
Comment	
Ref.	[Cha01] F. Chapoton, <i>Un endofoncteur de la catégorie des opérades</i> . Dialgebras & related operads, 105–110, Lect. Notes in Math., 1763, Springer, 2001.

Name	Dipterous algebra
Notation	$Dipt$
Def. oper. sym. rel.	$x * y, x \prec y$ $\begin{cases} (x * y) * z = x * (y * z) & \text{(associativity)} \\ (x \prec y) \prec z = x \prec (y * z) & \text{(dipterous relation)} \end{cases}$
Free alg.	$Dipt(V) = \bigoplus_{n \geq 1} (\mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n]) \otimes V^{\otimes n}$, for $n = 1$ the two copies of PT_1 are identified
rep. $\mathcal{P}(n)$	$Dipt(n) = (\mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n]) \otimes \mathbb{K}[\mathbb{S}_n], n \geq 2$, $Dipt_n = \mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n], n \geq 2$
$\dim \mathcal{P}(n)$	$1, 2 \times 2!, 6 \times 3!, 22 \times 4!, 90 \times 5!, 394 \times 6!, 1806 \times 7!, \dots, 2C_n \times n!, \dots$ where C_n is the Schröder number: $\sum_{n \geq 1} C_n t^n = \frac{1+t-\sqrt{1-6t+t^2}}{4}$
Gen. series	$f^{Dipt}(t) = \frac{1-t-\sqrt{1-6t+t^2}}{2}$
Dual operad	$Dipt^!$
Chain-cplx	Isomorphic to the total complex of a certain explicit bicomplex
Properties	binary, quadratic, ns, Koszul.
Alternative	“diptère” in French
Relationship	$Dend\text{-alg} \rightarrow Dipt\text{-alg}, \quad x * y := x \prec y + y \succ x$, a variation: replace the dipterous relation by $(x \prec y) * z + (x * y) \prec z = x \prec (y * z) + x * (y \prec z)$ to get <i>Hoch</i> -algebras, see [Ler11]. Same properties.
Unit	$1 \prec x = 0, x \prec 1 = x, \quad 1 * x = x = x * 1.$
Comment	dipterous = 2-fold in greek (the free algebra has 2 planar trees copies)
Ref.	[LR03] J.-L. Loday, M. Ronco, <i>Algèbres de Hopf colibres</i> , C.R.Acad. Sci. Paris t. 337, Ser. I (2003), 153 -158.

Name	Dual dipterous algebra
Notation	$Dipt^!$
Def. oper.	$x \dashv y, x * y$
sym.	
rel.	$\left\{ \begin{array}{lcl} (x * y) * z & = & x * (y * z), \\ (x \dashv y) \dashv z & = & x \dashv (y * z), \\ (x * y) \dashv z & = & 0, \\ (x \dashv y) * z & = & 0, \\ 0 & = & x * (y \dashv z) \\ 0 & = & x \dashv (y \dashv z). \end{array} \right.$
Free alg.	$Dipt^!(V) = \overline{T}(V) \oplus \overline{T}(V)$
rep. $\mathcal{P}(n)$	$Dipt^!(n) = \mathbb{K}^2 \otimes \mathbb{K}[S_n], \text{ } Dipt_n^! = \mathbb{K}^2, n \geq 2$
dim $\mathcal{P}(n)$	$1, 2 \times 2!, 2 \times 3!, 2 \times 4!, 2 \times 5!, 2 \times 6!, 2 \times 7!, \dots, 2 \times n!, \dots,$
Gen. series	$f^{Dipt^!}(t) = \frac{t+t^2}{1-t}$
Dual operad	$Dipt^{!!} = Dipt$
Chain-cplx	see ref.
Properties	binary, quadratic, ns, Koszul.
Alternative	
Relationship	$As\text{-alg} \rightarrow Dipt^!\text{-alg}, \quad x \dashv y := xy =: x \vdash y$
Unit	No
Comment	
Ref.	[LR03] J.-L. Loday, M. Ronco, <i>Algèbres de Hopf colibres</i> , C.R.Acad. Sci. Paris t. 337, Ser. I (2003), 153 -158.

Name	Two-associative algebra
Notation	$2as$
Def. oper. sym. rel.	$x * y, x \cdot y,$ $\begin{cases} (x * y) * z &= x * (y * z), \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z). \end{cases}$
Free alg.	$2as(V) = \bigoplus_{n \geq 1} (\mathbb{K}[T_n] \oplus \mathbb{K}[T_n]) \otimes V^{\otimes n}$, where T_n = planar trees for $n = 1$ the two copies of T_1 are identified
rep. $\mathcal{P}(n)$	$2as(n) = (\mathbb{K}[T_n] \oplus \mathbb{K}[T_n]) \otimes \mathbb{K}[\mathbb{S}_n]$, so $Dipt_n = \mathbb{K}[T_n] \oplus \mathbb{K}[T_n], n \geq 2$
$\dim \mathcal{P}(n)$	$1, 2 \times 2!, 6 \times 3!, 22 \times 4!, 90 \times 5!, 394 \times 6!, 1806 \times 7!, \dots, 2C_n \times n!, \dots$ where C_n is the Schröder number: $\sum_{n \geq 1} C_n t^n = \frac{1+t-\sqrt{1-6t+t^2}}{4}$
Gen. series	$f^{2as}(t) = \frac{1-t-\sqrt{1-6t+t^2}}{2}$
Dual operad	$2as^!$
Chain-cplx	Isomorphic to the total complex of a certain explicit bicomplex
Properties	binary, quadratic, ns, set-theoretic, Koszul.
Alternative	
Relationship	$2as\text{-alg} \rightarrow Dup\text{-alg}, \quad 2as\text{-alg} \rightarrow B_\infty\text{-alg}$
Unit	$1 \cdot x = x = x \cdot 1, \quad 1 * x = x = x * 1.$
Comment	
Ref.	[LR06] J.-L. Loday, M. Ronco, <i>On the structure of cofree Hopf algebras</i> , J. reine angew. Math. 592 (2006), 123–155.

Name	Dual 2-associative algebra
Notation	$2as^!$
Def. oper.	$x \cdot y, x * y$
sym.	
rel.	$\left\{ \begin{array}{lcl} (x * y) * z & = & x * (y * z), \\ (x \cdot y) \cdot z & = & x \cdot (y \cdot z), \\ (x \cdot y) * z & = & 0, \\ (x * y) \cdot z & = & 0, \\ 0 & = & x * (y \cdot z) \\ 0 & = & x \cdot (y * z). \end{array} \right.$
Free alg.	$2as^!(V) = V \oplus \oplus_{n \geq 2} V^{\otimes n} \oplus V^{\otimes n}$
rep. $\mathcal{P}(n)$	$Dipt^!(n) = \mathbb{K}[\mathbb{S}_n] \oplus \mathbb{K}[\mathbb{S}_n], \quad Dipt_n^! = \mathbb{K} \oplus \mathbb{K}, n \geq 2.$
$\dim \mathcal{P}(n)$	$1, 2 \times 2!, 2 \times 3!, 2 \times 4!, 2 \times 5!, 2 \times 6!, 2 \times 7!, \dots, 2 \times n!, \dots$
Gen. series	$f^{2as^!}(t) = \frac{t+t^2}{1-t}$
Dual operad	$(2as^!)^! = 2as$
Chain-cplx	see ref.
Properties	binary, quadratic, Koszul.
Alternative	
Relationship	
Unit	No
Comment	
Ref.	[LR06] J.-L. Loday, M. Ronco, <i>On the structure of cofree Hopf algebras</i> , J. reine angew. Math. 592 (2006), 123–155.

Name	Tridendriform algebra
Notation	<i>Tridend</i>
Def. oper. sym.	$x \prec y, x \succ y, x \cdot y$
rel.	$\left\{ \begin{array}{l} (x \prec y) \prec z = x \prec (y * z), \\ (x \succ y) \prec z = x \succ (y \prec z), \\ (x * y) \succ z = x \succ (y \succ z), \\ (x \succ y) \cdot z = x \succ (y \cdot z), \\ (x \prec y) \cdot z = x \cdot (y \succ z), \\ (x \cdot y) \prec z = x \cdot (y \prec z), \\ (x \cdot y) \cdot z = x \cdot (y \cdot z). \end{array} \right.$ <p>where $x * y := x \prec y + x \succ y + x \cdot y$.</p>
Free alg.	Planar rooted trees with variables in between the leaves
rep. $\mathcal{P}(n)$	$Tridend(n) = \mathbb{K}[PT_n] \otimes \mathbb{K}[\mathbb{S}_n], \quad Tridend_n = \mathbb{K}[PT_n]$
$\dim \mathcal{P}(n)$	$1, 3 \times 2!, 11 \times 3!, 45 \times 4!, 197 \times 5!, 903 \times 6!, \dots, C_n \times n!, \dots$ where C_n is the Schröder (or super Catalan) number
Gen. series	$f^{Tridend}(t) = \frac{-1+3t+\sqrt{1-6t+t^2}}{4t}$
Dual operad	$Triend^! = Trias$
Chain-cplx	Isomorphic to the total complex of a certain explicit tricomplex
Properties	binary, quadratic, ns, Koszul.
Alternative	sometimes called dendriform trialgebras
Relationship	$Tridend\text{-alg} \rightarrow As\text{-alg}, \quad x * y := x \prec y + y \succ x + x \cdot y,$ $ComTridend\text{-alg} \rightarrow Tridend\text{-alg}, \quad x \prec y := x \cdot y =: y \succ x$
Unit	$1 \prec x = 0, \quad x \prec 1 = x, \quad 1 \succ x = x, \quad x \succ 1 = 0, \quad 1 \cdot x = 0 = x \cdot 1.$
Comment	There exist several variations (see [Cha02] for instance).
Ref.	[LR04] J.-L. Loday and M. Ronco, <i>Trialgebras and families of polytopes</i> , Contemporary Mathematics (AMS) 346 (2004), 369–398.

Name	Triassociative algebra
Notation	<i>Trias</i>
Def. oper.	$x \dashv y, x \vdash y, x \perp y$ (no symmetry)
rel.	$\left\{ \begin{array}{l} (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (x \vdash y) \vdash z = x \vdash (y \perp z), \\ (x \dashv y) \dashv z = x \dashv (y \perp z), \\ (x \perp y) \dashv z = x \perp (y \dashv z), \\ (x \dashv y) \perp z = x \perp (y \vdash z), \\ (x \vdash y) \perp z = x \vdash (y \perp z), \\ (x \perp y) \vdash z = x \vdash (y \vdash z), \\ (x \perp y) \perp z = x \perp (y \perp z). \end{array} \right.$ <p>one relation for each cell of the pentagon.</p>
Free alg.	noncommutative polynomials with several variables marked
rep. $\mathcal{P}(n)$	$Trias(n) = \mathbb{K}^{2^n-1} \otimes \mathbb{K}[\mathbb{S}_n], \quad Trias_n = \mathbb{K}^{2^n-1}$
$\dim \mathcal{P}(n)$	$1, 3 \times 2!, 7 \times 3!, 15 \times 4!, 31 \times 5!, 63 \times 6!, 127 \times 7!, \dots, (2^n - 1) \times n!, \dots$
Gen. series	$f^{Trias}(t) = \frac{t}{(1-t)(1-2t)}$
Dual operad	$Trias^! = Tridend$
Properties	binary, quadratic, ns, set-theoretic, Koszul.
Alternative	Also called <i>associative trialgebra</i> , or for short, <i>trialgebra</i> .
Relationship	$As\text{-alg} \rightarrow Trias\text{-alg}, \quad x \dashv y = x \vdash y = x \perp y = xy$
Unit	Bar-unit: $x \dashv 1 = x = 1 \vdash x, 1 \dashv x = 0 = x \vdash 1, 1 \perp x = 0 = x \perp 1$
Comment	Relations easy to understand in terms of planar trees
Ref.	[LR04] J.-L. Loday and M. Ronco, <i>Trialgebras and families of polytopes</i> , Contemporary Mathematics (AMS) 346 (2004), 369–398.

Name	PostLie algebra
Notation	$x \circ y, [x, y]$
Def. oper.	$x \circ y, [x, y]$
sym.	$[x, y] = -[y, x]$
rel.	$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ $(x \circ y) \circ z - x \circ (y \circ z) - (x \circ z) \circ y + x \circ (z \circ y) = x \circ [y, z]$ $[x, y] \circ z = [x \circ z, y] + [x, y \circ z]$
Free alg.	$PostLie(V) \cong Lie(??(V))$
rep. $\mathcal{P}(n)$	
dim $\mathcal{P}(n)$	$1, 3, 20, 210, 3024, \dots, a(n), \dots$
Gen. series	$f(t) = -\log\left(\frac{1+\sqrt{1-4t}}{2}\right)$
Dual operad	$PostLie^! = ComTrias$
Chain-cplx	
Properties	binary, quadratic, Koszul.
Alternative	
Relationship	$PostLie\text{-alg} \rightarrow ??\text{-alg}, xy = x \circ y ???$ $PostLie\text{-alg} \rightarrow Lie\text{-alg}, \{x, y\} = x \circ y - y \circ x + [x, y]$ $PreLie\text{-alg} \rightarrow PostLie\text{-alg}, [x, y] = 0$
Unit	
Comment	
Ref.	[Val07] Vallette B., <i>Homology of generalized partition posets</i> , J. Pure Appl. Algebra 208 (2007), no. 2, 699–725.

Name	Commutative triassociative algebra
Notation	$ComTrias$
Def. oper.	$x \dashv y, x \perp y$
sym.	$x \perp y = y \perp x$
rel.	$\left\{ \begin{array}{l} (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (x \dashv y) \dashv z = x \dashv (z \dashv y), \\ (x \dashv y) \dashv z = x \dashv (y \perp z), \\ (x \perp y) \dashv z = x \perp (y \dashv z), \\ (x \perp y) \perp z = x \perp (y \perp z). \end{array} \right.$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t)$
Dual operad	$ComTrias^! = PostLie$
Chain-cplx	
Properties	binary, quadratic, Koszul.
Alternative	Triassociative with the following symmetry: $x \dashv y = y \vdash x$ and $x \perp y = y \perp x$
Relationship	$ComTrias\text{-alg} \rightarrow Perm\text{-alg}$
Unit	$x \dashv 1 = x, 1 \dashv x = 0, 1 \perp x = 0$
Comment	
Ref.	

Name	Commutative tridendriform algebra
Notation	CTD
Def. oper.	$x \prec y, x \cdot y$
sym.	$x \cdot y = y \cdot x$
rel.	$\left\{ \begin{array}{lcl} (x \prec y) \prec z & = & x \prec (y \prec z) + x \prec (z \prec y), \\ (x \cdot y) \prec z & = & x \cdot (y \prec z), \\ (x \prec z) \cdot y & = & x \cdot (y \prec z), \\ (x \cdot y) \cdot z & = & x \cdot (y \cdot z). \end{array} \right.$
Free alg.	$CTD(V)$ =quasi-shuffle algebra on $V = QSym(V)$
rep. $\mathcal{P}(n)$	$CTD(n) =$
$\dim \mathcal{P}(n)$	$1, 3, 13, 75, 541, 4683, \dots$
Gen. series	$f^{CTD}(t) = \frac{\exp(t)-1}{2-\exp(t)}$
Dual operad	$CTD^! =$
Chain-cplx	
Properties	binary, quadratic, Koszul.
Alternative	Handy to introduce $x * y := x \prec y + y \prec x + x \cdot y$ (assoc. and comm.) equivalently: tridendriform with symmetry: $x \prec y = y \succ x, x \cdot y = y \cdot x$
Relationship	$CTD\text{-alg} \rightarrow Tridend\text{-alg}$
Unit	$1 \prec x = 0, x \prec 1 = x, 1 \cdot x = 0 = x \cdot 1.$
Comment	
Ref.	[Lod07] J.-L. Loday, <i>On the algebra of quasi-shuffles</i> , Manuscripta mathematica 123 (1), (2007), 79–93.

Name	Dual CTD algebra
Notation	$CTD^!$
Def. oper.	$x \dashv y, [x, y]$
sym.	$[x, y] = -[y, x]$
rel.	
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 3, \dots$
Gen. series	$f^{CTD^!}(t) =$
Dual operad	$(CTD^!)^! = CTD$
Chain-cplx	
Properties	binary, quadratic, Koszul.
Alternative	
Relationship	$Trias\text{-alg} \rightarrow CTD^!\text{-alg}$
Unit	
Comment	
Ref.	

Name	Gerstenhaber algebra
	underlying objects: graded vector spaces
Notation	<i>Gerst</i>
Def. oper.	m = binary operation of degree 0, c = binary operation of degree 1
sym.	m symmetric, c antisymmetric
rel.	$c \circ_1 c + (c \circ_1 c)^{(123)} + (c \circ_1 c)^{(321)} = 0,$ $c \circ_1 m - m \circ_2 c - (m \circ_1 c)^{(23)} = 0,$ $m \circ_1 m - m \circ_2 m = 0 .$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	binary, quadratic, Koszul.
Alternative	
Relationship	
Unit	
Comment	To get the relations in terms of elements, do not forget to apply the Koszul sign rule. Observe that the last relation is associativity.
Ref.	

Name	Batalin-Vilkovisky algebra
	underlying objects: graded vector spaces
Notation	BV -alg
Def. oper.	Δ unary degree 1, m binary degree 0, c binary degree 1
sym.	m symmetric, c antisymmetric
rel.	$m \circ_1 m - m \circ_2 m = 0,$ $\Delta^2 = 0,$ $c = \Delta \circ_1 m + m \circ_1 \Delta + m \circ_2 \Delta,$ $c \circ_1 c + (c \circ_1 c)^{(123)} + (c \circ_1 c)^{(321)} = 0,$ $c \circ_1 m - m \circ_2 c - (m \circ_1 c)^{(23)} = 0,$ $\Delta \circ_1 c + c \circ_1 \Delta + c \circ_2 \Delta = 0.$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	unary and binary, inhomogeneous quadratic, Koszul.
Alternative	Generated by Δ and m only.
Relationship	BV -alg \rightarrow $Gerst$ -alg
Unit	
Comment	Defined by J.-L. Koszul. To get the relations in terms of elements, do not forget to apply the Koszul sign rule.
Ref.	[GCTV09]

Name	Magmatic algebra
Notation	Mag
Def. oper. sym. rel.	xy
Free alg.	any parenthesizing of words
rep. $\mathcal{P}(n)$	$Mag(n) = \mathbb{K}[PBT_n] \otimes \mathbb{K}[\mathbb{S}_n]$, $Mag_n = \mathbb{K}[PBT_n]$ (planar binary trees with n leaves)
$\dim \mathcal{P}(n)$	$1, 1 \times 2!, 2 \times 3!, 5 \times 4!, 14 \times 5!, 42 \times 6!, 132 \times 7!, \dots, c_{n-1} \times n!, \dots$ where $c_n = \frac{1}{n+1} \binom{2n}{n}$ (Catalan number)
Gen. series	$f^{Mag}(t) = (1/2)(1 - \sqrt{1 - 4t})$
Dual operad	$Mag^! = Nil_2$
Chain-cplx	
Properties	binary, quadratic, ns, set-theoretic, Koszul.
Alternative	sometimes (improperly) called <i>nonassociative algebra</i> .
Relationship	many “inclusions” (all types of alg. with only one gen. op.)
Unit	$1x = x = x1$
Comment	
Ref.	

Name	2-Nilpotent algebra
Notation	Nil_2
Def. oper.	xy
sym.	
rel.	$(xy)z = 0 = x(yz)$
Free alg.	$Nil_2(V) = V \oplus V^{\otimes 2}$
rep. $\mathcal{P}(n)$	$Nil_2(2) = \mathbb{K}[\mathbb{S}_2]$, $Nil_2(n) = 0$ for $n \geq 3$
$\dim \mathcal{P}(n)$	$1, 2, 0, 0, 0, 0, 0, \dots$
Gen. series	$f^{\mathcal{P}}(t) = t + t^2$
Dual operad	$Nil_2^! = Mag$
Chain-cplx	
Properties	binary, quadratic, ns, Koszul.
Alternative	
Relationship	
Unit	no
Comment	
Ref.	

Name	Commutative magmatic algebra
Notation	$ComMag$
Def. oper.	$x \cdot y$
sym.	$x \cdot y = y \cdot x$
rel.	none
Free alg.	any parenthesizing of commutative words
rep. $\mathcal{P}(n)$	$ComMag(n) = \mathbb{K}[shBT(n)]$ $shBT(n) = \{\text{shuffle binary trees with } n \text{ leaves}\}$
$\dim \mathcal{P}(n)$	$\dim ComMag(n) = (2n - 3)!! = 1 \times 3 \times \cdots \times (2n - 3)$
Gen. series	$f^{ComMag}(t) =$
Dual operad	$ComMag^1\text{-alg}$: $[x, y]$ antisymmetric, $[[x, y], z] = 0$.
Chain-cplx	
Properties	binary, quadratic, set-theoretic, Koszul.
Alternative	
Relationship	$ComMag \hookrightarrow PreLie, x \cdot y := \{x, y\} + \{y, x\}$, cf. [BL11]
Unit	$1x = x = x1$
Comment	
Ref.	

Name **Anti-symmetric nilpotent algebra**

Notation $ComMag^!$

Def. oper. $x \cdot y$
 sym. $x \cdot y = -y \cdot x$
 rel. $(xy)z = 0$

Free alg.

rep. $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$

Gen. series

Dual operad $ComMag.$

Chain-cplx

Properties binary, quadratic, set-theoretic, Koszul.

Alternative

Relationship

Unit

Comment

Ref.

Name **Quadri-algebra**

Notation *Quadri*

Def. oper. $x \nwarrow y, x \nearrow y, x \searrow y, x \swarrow y$ called NW, NE, SE, SW oper.
rel.

$$\begin{aligned}
(x \nwarrow y) \nwarrow z &= x \nwarrow (y \star z) & (x \nearrow y) \nwarrow z &= x \nearrow (y \prec z) & (x \wedge y) \nearrow z &= x \nearrow (y \succ z) \\
(x \swarrow y) \nwarrow z &= x \swarrow (y \wedge z) & (x \searrow y) \nwarrow z &= x \searrow (y \nwarrow z) & (x \vee y) \nearrow z &= x \searrow (y \nearrow z) \\
(x \prec y) \swarrow z &= x \swarrow (y \vee z) & (x \succ y) \swarrow z &= x \searrow (y \swarrow z) & (x \star y) \searrow z &= x \searrow (y \searrow z)
\end{aligned}$$

where

$$\begin{aligned}
x \succ y &:= x \nearrow y + x \searrow y, & x \prec y &:= x \nwarrow y + x \swarrow y \\
x \vee y &:= x \searrow y + x \swarrow y, & x \wedge y &:= x \nearrow y + x \nwarrow y \\
x \star y &:= x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y \\
&= x \succ y + x \prec y = x \vee y + x \wedge y
\end{aligned}$$

Free alg.

rep. $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$ $1, 4 \times 2!, 23 \times 3!, 156 \times 4!, 1162 \times 5!, 9192 \times 6!, \dots$

$$\dim \mathcal{P}_n = \frac{1}{n} \sum_{j=n}^{2n-1} \binom{3n}{n+1+j} \binom{j-1}{j-n}$$

Gen. series $f(t)$

Dual operad *Quadri*[!]

Properties binary, quadratic, ns, Koszul.

Alternative $Quadri = Dend \blacksquare Dend = PreLie \bullet Dend$

Relationship Related to dendriform in several ways

Unit partial unit (like in dendriform)

Comment There exist several variations like $\mathcal{P} \blacksquare \mathcal{P}$ (cf. Ph. Leroux [Ler04])

Ref. [AL04] M. Aguiar, J.-L. Loday, *Quadri-algebras*, J. Pure Applied Algebra 191 (2004), 205–221.

Name	Dual quadri-algebra
Notation	$Quad^!$
Def. oper.	$x \nearrow y, x \nearrow y, x \searrow y, x \swarrow y$
sym.	
rel.	A FAIRE
Free alg.	
rep.	$\mathcal{P}(n)$
dim $\mathcal{P}(n)$	$1, 4 \times 2!, 9 \times 3!, 16 \times 4!, 25 \times 5!, \dots, n^2 \times n!, \dots$
Gen. series	$f(t)$
Dual operad	
Chain-cplx	
Properties	binary, quadratic, Koszul.
Alternative	
Relationship	
Unit	
Comment	
Ref.	[Val08] B.Vallette, <i>Manin products, Koszul duality, Loday algebras and Deligne conjecture</i> . J. Reine Angew. Math. 620 (2008), 105–164.

Name	Duplicial algebra
Notation	Dup
Def. oper. sym.	$x \prec y, x \succ y$
rel.	$\begin{cases} (x \prec y) \prec z &= x \prec (y \prec z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x \succ y) \succ z &= x \succ (y \succ z). \end{cases}$
Free alg.	$Dup(V) = \oplus_{n \geq 1} \mathbb{K}[PBT_{n+1}] \otimes V^{\otimes n}$, for p.b. trees s and t : $x \succ y$ over operation is grafting of x on the leftmost leaf of y $x \prec y$ under operation is grafting of y on the rightmost leaf of x
rep. $\mathcal{P}(n)$	$Dup(n) = \mathbb{K}[PBT_{n+1}] \otimes \mathbb{K}[\mathbb{S}_n]$, so $Dup_n = \mathbb{K}[PBT_{n+1}]$
dim $\mathcal{P}(n)$	$1, 2 \times 2!, 5 \times 3!, 14 \times 4!, 42 \times 5!, 132 \times 6!, 429 \times 7!, \dots, c_n \times n!, \dots$ where $c_n = \frac{1}{n+1} \binom{2n}{n}$ is the Catalan number
Gen. series	$f^{Dup}(t) = \frac{1-2t-\sqrt{1-4t}}{2t} = y, \quad y^2 - (1-2t)y + t = 0$
Dual operad	$Dup^!$
Chain-cplx	Isomorphic to the total complex of a certain explicit bicomplex
Properties	binary, quadratic, ns, set-theoretic, Koszul.
Alternative	
Relationship	$2as\text{-alg} \rightarrow Dup\text{-alg} \rightarrow As^2\text{-alg}$
Unit	
Comment	The associator of $xy := x \succ y - x \prec y$ is $as(x, y, z) = x \prec (y \succ z) - (x \prec y) \succ z$. Appeared first in [BF03]
Ref.	[Lod08] J.-L. Loday, <i>Generalized bialgebras and triples of operads</i> , Astérisque (2008), no 320, x+116 p.

Name	Dual duplicial algebra
Notation	$Dup^!$
Def. oper.	$x \prec y, x \succ y$
sym.	
rel.	$\left\{ \begin{array}{lcl} (x \prec y) \prec z & = & x \prec (y \prec z), \\ (x \prec y) \succ z & = & 0, \\ (x \succ y) \prec z & = & x \succ (y \prec z), \\ 0 & = & x \prec (y \succ z), \\ (x \succ y) \succ z & = & x \succ (y \succ z). \end{array} \right.$
Free alg.	$Dup^!(V) = \oplus_{n \geq 1} n V^{\otimes n}$ noncommutative polynomials with one variable marked
rep. $\mathcal{P}(n)$	$Dup^!(n) = \mathbb{K}^n \otimes \mathbb{K}[\mathbb{S}_n], \quad Dup_n^! = \mathbb{K}^n$
$\dim \mathcal{P}(n)$	$1, 2 \times 2!, 3 \times 3!, 4 \times 4!, 5 \times 5!, 6 \times 6!, 7 \times 7!, \dots, n \times n!, \dots,$
Gen. series	$f^{Dup^!}(t) = \frac{t}{(1-t)^2}$
Dual operad	$Dup^{!!} = Dup$
Chain-cplx	see ref.
Properties	binary, quadratic, Koszul.
Alternative	
Relationship	
Unit	
Comment	
Ref.	

Name	$As^{(2)}$ - algebra
Notation	$As^{(2)}$
Def. oper.	$x * y, x \cdot y$
sym.	
rel.	$(x \circ_1 y) \circ_2 z = x \circ_1 (y \circ_2 z)$ any values (4 relations)
Free alg.	
rep. $\mathcal{P}(n)$	$As_n^{(2)} = \mathbb{K}[\{0, 1\}^{n-1}]$
$\dim \mathcal{P}(n)$	2^{n-1}
Gen. series	$f^{As^{(2)}}(t) = \frac{t}{1-2t}$
Dual operad	$As^{(2)\dagger} = As^{(2)}$
Chain-cplx	
Properties	binary, quadratic, set-theoretic, ns.
Alternative	
Relationship	$2as\text{-alg} \rightarrow Dup\text{-alg} \rightarrow As^{(2)}\text{-alg}$
Unit	
Comment	Variations: $As^{(k)}$
Ref.	

Name	$As^{\langle 2 \rangle}$ -algebra
Notation	$As^{\langle 2 \rangle}$, compatible products algebra
Def. oper.	$x * y, x \cdot y$
sym.	
rel.	$(x * y) * z = x * (y * z)$ $(x * y) \cdot z + (x \cdot y) * z = x * (y \cdot z) + x \cdot (y * z)$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
Free alg.	similar to dendriform
rep. \mathcal{P}_n	$As_n^{\langle 2 \rangle} = \mathbb{K}[PBT_{n+1}]$
$\dim \mathcal{P}_n$	$c_n = \frac{1}{n+1} \binom{2n}{n}$
Gen. series	$f^{As^{\langle 2 \rangle}}(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$
Dual operad	An $(As^{\langle 2 \rangle})^!$ -algebra has 2 generating binary operations and 5 relations: $(x \circ_i y) \circ_j z = x \circ_i (y \circ_j z)$ and $(x * y) \cdot z + (x \cdot y) * z = x * (y \cdot z) + x \cdot (y * z)$
Chain-cplx	
Properties	binary, quadratic, ns, Koszul.
Alternative	
Relationship	$(As^{\langle 2 \rangle})^!-\mathbf{alg} \rightarrow As^{\langle 2 \rangle}-\mathbf{alg} \rightarrow As^{\langle 2 \rangle}-\mathbf{alg}$
Unit	
Comment	equivalently $\lambda x * y + \mu x \cdot y$ is associative for any λ, μ Variations: $As^{\langle k \rangle}$, Hoch-alg.
Ref.	for instance [Gon05], [OS06], [Dot09].

Name	L-dendriform algebra
Notation	$L\text{-dend}$
Def. oper.	$x \triangleright y$ and $x \triangleleft y$
sym.	
rel.	$x \triangleright (y \triangleright z) - (x \bullet y) \triangleright z = y \triangleright (x \triangleright z) - (y \bullet x) \triangleright z$ $x \triangleright (y \triangleleft z) - (x \triangleright y) \triangleleft z = y \triangleleft (x \bullet z) - (y \triangleleft x) \triangleleft z$ <p>where $x \bullet y := x \triangleright y + x \triangleleft y$.</p>
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	$L\text{-dend} = \text{PreLie} \bullet \text{PreLie} ?$
Alternative	
Relationship	$Dend\text{-alg} \rightarrow L\text{-dend}\text{-alg}$ via $x \triangleright y := x \succ y$ and $x \triangleleft y := x \prec y$ $L\text{-dend}\text{-alg} \rightarrow \text{PreLie}\text{-alg}, (A, \triangleright, \triangleleft) \mapsto (A, \bullet)$
Unit	
Comment	Various variations like $L\text{-quad}\text{-alg}$ [Bai10]
Ref.	Bai C.M., Liu L.G., Ni X., <i>L-dendriform algebras</i> , preprint (2010).

Name	Lie-admissible algebra
Notation	$Lie\text{-}adm$
Def. oper.	xy
sym.	
rel.	$[x, y] = xy - yx$ is a Lie bracket, that is $\sum_{\sigma} \text{sgn}(\sigma) \sigma((xy)z - x(yz)) = 0$
Free alg.	
rep. $\mathcal{P}(n)$	$Lie\text{-}adm(n) = ?$
$\dim \mathcal{P}(n)$	$1, 2, 11, \dots$
Gen. series	$f^{Lie\text{-}adm}(t) = ?$ its dimensions are $1, 2, 1, ??$
Dual operad	$Lie\text{-}adm^!$
Chain-cplx	
Properties	binary, quadratic, Koszul ??.
Alternative	
Relationship	$As\text{-}alg \rightarrow PreLie\text{-}alg \rightarrow Lie\text{-}adm\text{-}alg$
Unit	
Comment	
Ref.	

Name	PreLiePerm algebra
Notation	$PreLiePerm$
Def. oper. sym.	$x \prec y, x \succ y, x * y$
rel.	$\left\{ \begin{array}{lcl} (x \prec y) \prec z & = & x \prec (y * z), \\ (x \prec y) \prec z & = & x \prec (z * y), \\ (x * y) \succ z & = & x \succ (y \succ z), \\ (x \succ y) \prec z & = & x \succ (z \succ y), \\ (x \succ y) \prec z & = & x \succ (y \prec z), \end{array} \right.$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	1, 6,
Gen. series	
Dual operad	$PermPreLie = Perm \circ PreLie = Perm \otimes PreLie$
Chain-cplx	
Properties	binary, quadratic, set-theoretic.
Alternative	
Relationship	$PreLiePerm = PreLie \bullet Perm$, see [Val08] $Zinb\text{-}alg \rightarrow PreLiePerm\text{-}alg \rightarrow Dend\text{-}alg$
Unit	no
Comment	
Ref.	[Val08] B.Vallette, <i>Manin products, Koszul duality, Loday algebras and Deligne conjecture</i> . J. Reine Angew. Math. 620 (2008), 105–164.

Name	Alternative algebra
Notation	<i>Altern</i>
Def. oper.	xy
sym.	
rel.	$(xy)z - x(yz) = -(yx)z + y(xz),$ $(xy)z - x(yz) = -(xz)y + x(zy),$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	1, 2, 7, 32, 175, ??
Gen. series	$f(t)$
Dual operad	$(xy)z = x(yz),$ $xyz + yxz + zxy + xzy + yzx + zyx = 0$ $\dim \text{Altern}^!(n) = 1, 2, 5, 12, 15, \dots$
Chain-cplx	
Properties	binary, quadratic, nonKoszul [DZ09].
Alternative	Equivalent presentation: the associator $as(x, y, z)$ is skew-symmetric: $\sigma \cdot as(x, y, z) = \text{sgn}(\sigma)as(x, y, z)$
Relationship	<i>As</i> -alg \rightarrow <i>Altern</i> -alg
Unit	$1x = x = x1$
Comment	The octonions are an example of alternative algebra Integration: Moufang loops $\dim \mathcal{P}(n)$ computed by W. Moreira
Ref.	[She04] Shestakov, I. P., <i>Moufang loops and alternative algebras.</i> Proc. Amer. Math. Soc. 132 (2004), no. 2, 313–316.

Name	Parametrized-one-relation algebra
Notation	<i>Param1rel</i>
Def. oper.	xy
sym.	none
rel.	$(xy)z = \sum_{\sigma \in \mathbb{S}_3} a_\sigma \sigma \cdot x(yz)$ where $a_\sigma \in \mathbb{K}$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t) =$
Dual operad	$x(yz) = \sum_{\sigma \in \mathbb{S}_3} \text{sgn}(\sigma) a_\sigma \sigma^{-1} \cdot (xy)z$
Chain-cplx	
Properties	
Alternative	
Relationship	Many classical examples are particular case: <i>As, Leib, Zinb, Pois</i>
Unit	
Comment	Problem: for which families of parameters $\{a_\sigma\}_{\sigma \in \mathbb{S}_3}$ is the operad a Koszul operad ?
Ref.	

Name	Magmatic-Fine algebra
Notation	$MagFine$
Def. oper. sym. rel.	$(x_1, \dots, x_n)_i^n$ for $1 \leq i \leq n-2, n \geq 3$
Free alg.	Described in terms of some coloured planar rooted trees
rep. $\mathcal{P}(n)$	a sum of regular representations indexed as said above
$\dim \mathcal{P}(n)$	$1, 0, 1 \times 3!, 2 \times 4!, 6 \times 5!, 18 \times 6!, 57 \times 7!, \dots, F_{n-1} \times n!, \dots$ where $F_n = \text{Fine number}$
Gen. series	$f^{MagFine}(t) = \frac{1+2t-\sqrt{1-4t}}{2(2+t)}$
Dual operad	$MagFine^!$ same generating operations, any composition is trivial $\dim MagFine_n^! = n-2$ $f^{MagFine^!}(t) = t + \frac{t^3}{(1-t)^2}$
Chain-cplx	
Properties	multi-ary, quadratic, ns, Koszul.
Alternative	
Relationship	
Unit	
Comment	
Ref.	[HLR08] Holtkamp, R., Loday, J.-L., Ronco, M., <i>Coassociative magmatic bialgebras and the Fine numbers</i> , J.Alg.Comb. 28 (2008), 97–114.

Name	Generic magmatic algebra
Notation	$GenMag$
Def. oper. sym. rel.	a_n generating operations of arity n , $a_1 = 1$
Free alg.	
rep. $\mathcal{P}(n)$	a sum of regular representations
$\dim \mathcal{P}(n)$	
Gen. series	$f^{GenMag}(t) = \sum_n b_n t^n$, $b_n =$ polynomial in a_1, \dots, a_n
Dual operad	same generating operations, any composition is trivial $\dim GenMag_n^! = a_n$, $f^{GenMag^!}(t) = \sum_n a_n t^n$
Chain-cplx	
Properties	multi-ary, quadratic, ns, Koszul.
Alternative	
Relationship	For $MagFine^!$ $a_n = n - 2$.
Unit	
Comment	give a nice proof of the inversion formula for a generic power series (computation of the polynomial b_n)
Ref.	[Lod05] Loday, J.-L., <i>Inversion of integral series enumerating planar trees</i> . Séminaire lotharingien Comb. 53 (2005), exposé B53d, 16pp.

Name	Nonassociative permutative algebra
Notation	NAP
Def. oper.	xy
sym.	
rel.	$(xy)z = (xz)y$
Free alg.	$NAP(V)$ can be described in terms of rooted trees
rep. $\mathcal{P}(n)$	$NAP(n) = PreLie(n)$ as \mathbb{S}_n -modules
$\dim \mathcal{P}(n)$	$1, 2, 9, 64, 625, 1296, 117649, \dots, n^{n-1}, \dots$
Gen. series	$f^{NAP}(t) = y$ which satisfies $y = t \exp(y)$
Dual operad	$NAP^!$
Chain-cplx	
Properties	binary, quadratic, set-theoretic, Koszul.
Alternative	
Relationship	$Perm\text{-}alg \rightarrow NAP\text{-}alg$
Unit	no
Comment	This is right NAP algebra
Ref.	[Liv06] Livernet, M., <i>A rigidity theorem for pre-Lie algebras</i> , J. Pure Appl. Algebra 207 (2006), no. 1, 1–18.

Name	Moufang algebra
Notation	<i>Moufang</i>
Def. oper.	xy
sym.	
rel.	$x(yz) + z(yx) = (xy)z + (zy)x$ $((xy)z)t + ((zy)x)t = x(y(zt)) + z(y(xt)),$ $t(x(yz) + z(yx)) = ((tx)y)z + ((tz)y)x,$ $(xy)(tz) + (zy)(tx) = (x(yt))z + (z(yt))x$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	1, 2, 7, 40, ??
Gen. series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	binary
Alternative	Relation sometimes written in terms of the Jacobiator
Relationship	$Altern\text{-}alg \xrightarrow{-} Moufang\text{-}alg \rightarrow NC\text{Jordan}\text{-}alg$
Unit	
Comment	Integration: Moufang loop. From this presentation there is an obvious definition of “nonantisymmetric Malcev algebra”.
Ref.	[PIS04] Shestakov, I., Pérez-Izquierdo, J.M., <i>An envelope for Malcev algebras</i> . J. Alg. 272 (2004), 379–393.

Name	Malcev algebra
Notation	<i>Malcev</i>
Def. oper.	xy
sym.	$xy = -yx$
rel.	$((xy)z)t + (x(yz))t + x((yz)t) + x(y(z t)) = (xy)(zt)$
Free alg.	
rep.	$\mathcal{P}(n)$
dim $\mathcal{P}(n)$	1, 1, 3, 9, ??
Gen. series	$f(t)$
Dual operad	
Chain-cplx	
Properties	cubic.
Alternative	
Relationship	$Altern\text{-}alg \xrightarrow{-} Malcev\text{-}alg, \quad Lie\text{-}alg \rightarrow Malcev\text{-}alg$
Unit	
Comment	
Ref.	[PIS04] Shestakov, I., Pérez-Izquierdo, J.M., <i>An envelope for Malcev algebras</i> . J. Alg. 272 (2004), 379–393.

Name	Novikov algebra
Notation	<i>Novikov</i>
Def. oper.	xy
sym.	
rel.	$(xy)z - x(yz) = (xz)y - x(zy)$ $x(yz) = y(xz)$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	1, 2,
Gen. series	$f^{Novikov}(t) =$
Dual operad	
Chain-cplx	
Properties	binary, quadratic.
Alternative	Novikov is pre-Lie + $x(yz) = y(xz)$
Relationship	<i>Novikov</i> -alg \rightarrow <i>PreLie</i> -alg
Unit	
Comment	
Ref.	

Name	Double Lie algebra
Notation	<i>DoubleLie</i>
Def. oper.	$[x, y], \{x, y\}$
sym.	$[x, y] = -[y, x], \{x, y\} = \{y, x\}$
rel.	Any linear combination is a Lie bracket
Free alg.	
rep.	$\mathcal{P}(n)$
dim $\mathcal{P}(n)$	
Gen. series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	binary, quadratic.
Alternative	
Relationship	
Unit	
Comment	
Ref.	[DK07] Dotsenko V., Khoroshkin A., <i>Character formulas for the operad of two compatible brackets and for the bihamiltonian operad</i> , Functional Analysis and Its Applications, 41 (2007), no.1, 1-17.

Name	DipreLie algebra
Notation	<i>DipreLie</i>
Def. oper.	$x \circ y, x \bullet y$
sym.	
rel.	$(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y)$ $(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y)$ $(x \circ y) \bullet z - x \circ (y \bullet z) = (x \bullet z) \circ y - x \bullet (z \circ y)$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	binary, quadratic.
Alternative	
Relationship	relationship with the Jacobian conjecture (T. Maszczyk)
Unit	
Comment	
Ref.	Maszczyk T., unpublished.

Name	Akivis algebra
Notation	<i>Akivis</i>
Def. oper.	$[x, y], (x, y, z)$
sym.	$[x, y] = -[y, x]$
rel.	$[[x, y], z] + [[y, z], x] + [[z, x], y] =$ $(x, y, z) + (y, z, x) + (z, x, y) - (x, z, y) - (y, x, z) - (z, y, x)$ (Akivis relation)
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 1, 8, \dots,$
Gen. series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	binary and ternary, quadratic.
Alternative	relation also called “nonassociative Jacobi identity”
Relationship	<i>Akivis</i> -alg \rightarrow <i>Sabinin</i> -alg, <i>Mag</i> -alg \rightarrow <i>Akivis</i> -alg, $[x, y] = xy - yx$, $(x, y, z) = (xy)z - x(yz)$
Unit	
Comment	
Ref.	[BHP05]

Name	Sabinin algebra
Notation	<i>Sabinin</i>
Def. oper.	$\langle x_1, \dots, x_m; y, z \rangle, m \geq 0$ $\Phi(x_1, \dots, x_m; y_1, \dots, y_n), \quad m \geq 1, n \geq 2,$
sym.	$\langle x_1, \dots, x_m; y, z \rangle = -\langle x_1, \dots, x_m; z, y \rangle$ $\Phi(x_1, \dots, x_m; y_1, \dots, y_n) = \Phi(\omega(x_1, \dots, x_m); \theta(y_1, \dots, y_n)), \omega \in \mathbb{S}_m, \theta \in \mathbb{S}_n$
rel.	$\langle x_1, \dots, x_r, u, v, x_{r+1}, \dots, x_m; y, z \rangle - \langle x_1, \dots, x_r, v, u, x_{r+1}, \dots, x_m; y, z \rangle$ $+ \sum_{k=0}^r \sum_{\sigma} \langle x_{\sigma(1)}, \dots, x_{\sigma(k)}; \langle x_{\sigma(k+1)}, \dots, x_{\sigma(r)}; u, v \rangle, x_{r+1}, \dots, x_m; y, z \rangle$ where σ is a $(k, r-k)$ -shuffle $K_{u,v,w} [\langle x_1, \dots, x_r; y, z \rangle + \sum_{k=0}^r \sum_{\sigma} \langle x_{\sigma(1)}, \dots, x_{\sigma(k)}; \langle x_{\sigma(k+1)}, \dots, x_{\sigma(r)}; v, w \rangle, u \rangle] = 0$ where $K_{u,v,w}$ is the sum over all cyclic permutations
Free alg.	
rep. $\mathcal{P}(n)$	
dim $\mathcal{P}(n)$	$1, 1, 8, 78, 1104, \dots$
Gen. series	$f^{Sab}(t) = \log(1 + (1/2)(1 - \sqrt{1 - 4t}))$
Dual operad	
Chain-cplx	
Properties	quadratic, ns.
Alternative	There exists a more compact form of the relations which uses the tensor algebra over the Sabinin algebra
Relationship	$Mag\text{-}alg \longrightarrow Sabinin\text{-}alg, \langle y, z \rangle = yz - zy, \langle x; y, z \rangle = ??$ $Akivis\text{-}alg \longrightarrow Sabinin\text{-}alg, \langle y, z \rangle = -[y, z],$ $\langle x; y, z \rangle = (x, z, y) - (x, y, z), \langle x_1, \dots, x_m; y, z \rangle = 0, m \geq 2$
Unit	
Comment	Integration: local analytic loop
Ref.	[PI07] D. Pérez-Izquierdo, <i>Algebras, hyperalgebras, nonassociative bialgebras and loops</i> . Adv. in Maths 208 (2007), 834–876.

Name	Jordan triples
Notation	JT
Def. oper.	(xyz) or (x, y, z)
sym.	$(xyz) = (zyx)$
rel.	$(xy(ztu)) = ((xyz)tu) - (z(txy)u) + (zt(xyu))$
Free alg.	
rep. $\mathcal{P}(2n-1)$	
$\dim \mathcal{P}(2n-1)$	1, 3, 50, ??
Gen. series	$f(t)$
Dual operad	
Chain-cplx	
Properties	ternary, quadratic, Koszul ?.
Alternative	
Relationship	
Unit	
Comment	Remark that the quadratic relation, as written here, has a Leibniz flavor. Dimension computed by Walter Moreira
Ref.	

Name	Totally associative ternary algebra
Notation	$t\text{-}As^{\langle 3 \rangle}$
Def. oper.	(xyz)
sym.	
rel.	$((xyz)uv) = (x(yzu)v) = (xy(zuv))$
Free alg.	
rep.	$\mathcal{P}(2n-1)$
dim $\mathcal{P}(2n-1)$	$1, 3!, 5!, \dots, (2n+1)!, \dots$
Gen. series	$f^{t\text{-}As^3}(t) = \frac{t}{1-t^2}$
Dual operad	$t\text{-}As^{\langle 3 \rangle}{}^! = p\text{-}As^{\langle 3 \rangle}$
Chain-cplx	
Properties	ternary, quadratic, ns, set-theoretic, Koszul ?.
Alternative	
Relationship	$As\text{-alg} \rightarrow t\text{-}As^{\langle 3 \rangle}$
Unit	
Comment	
Ref.	[Gne97] Gnedbaye, A.V., <i>Opérades des algèbres $(k+1)$-aires</i> . Operads: Proceedings of Renaissance Conferences, 83–113, Contemp. Math., 202, Amer. Math. Soc., Providence, RI, 1997.

Name	Partially associative ternary algebra
Notation	$p\text{-}As^{\langle 3 \rangle}$
Def. oper.	(xyz)
sym.	
rel.	$((xyz)uv) + (x(yzu)v) + (xy(zuv)) = 0$
Free alg.	
rep.	$\mathcal{P}(2n - 1)$
dim $\mathcal{P}(2n - 1)$	
Gen. series	$f(t)$
Dual operad	$p\text{-}As^{\langle 3 \rangle}{}^! = t\text{-}As^{\langle 3 \rangle}$
Chain-cplx	
Properties	ternary, quadratic, ns, Koszul ?.
Alternative	
Relationship	
Unit	
Comment	
Ref.	[Gne97] Gnedbaye, A.V., <i>Opérades des algèbres $(k + 1)$-aires</i> . Operads: Proceedings of Renaissance Conferences, 83–113, Contemp. Math., 202, Amer. Math. Soc., Providence, RI, 1997.

Name	Lie triple systems
Notation	LTS
Def. oper.	$[xyz]$
sym.	$[xyz] = -[yxz]$
	$[xyz] + [yzx] + [zxy] = 0$
rel.	$[xy[ztu]] = [[xyz]tu] - [z[txy]u] + [zt[xyu]]$
Free alg.	
rep.	$\mathcal{P}(n)$
dim $\mathcal{P}(n)$	
Gen. series	$f(t)$
Dual operad	$LT^!$
Chain-cplx	
Properties	ternary, quadratic, Koszul ?.
Alternative	the relation admits many different versions due to the symmetry
Relationship	$Lie\text{-alg} \rightarrow LTS\text{-alg}, [xyz] = [[xy]z]$
Comment	Appreciate the Leibniz presentation Integration: symmetric spaces
Ref.	[Loo69] Loos O., Symmetric spaces. I. General theory. W. A. Benjamin, Inc., New York-Amsterdam (1969) viii+198 pp.

Name	Lie-Yamaguti algebra
Notation	<i>Lie – Yamaguti</i>
Def. oper.	$x \cdot y, \quad [x, y, z]$
sym.	$x \cdot y = -y \cdot x, \quad [x, y, z] = -[y, x, z]$
	$[x, y, z] + [y, z, x] + [z, x, y] + (x \cdot y) \cdot z + (y \cdot z) \cdot x + (z \cdot x) \cdot y = 0$
rel.	$\sum_{cyclic} [x \cdot y, z, t] = 0$
	$[x, y, u \cdot v] = u \cdot [x, y, v] + [x, y, u] \cdot v$
	$[x, y, [z, t, u]] = [[x, y, z], t, u] - [z, [t, x, y], u] + [z, t, [x, y, u]]$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t)$
Dual operad	$LT^!$
Chain-cplx	
Properties	binary and ternary, quadratic, Koszul ?.
Alternative	Generalized Lie triple systems
Relationship	$LTS\text{-alg} \rightarrow LY\text{-alg}, x \cdot y = 0, [x, y, z] = [xyz]$
Unit	
Comment	
Ref.	[KW01] R. Kinyon, M. Weinstein, A., <i>Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces</i> . Amer. J. Math. 123 (2001), no. 3, 525–550.

Name	Interchange algebra
Notation	<i>Interchange</i>
Def. oper.	$x \cdot y, \quad x * y$
sym.	
rel.	$(x \cdot y) * (z \cdot t) = (x * z) \cdot (y * t)$
Free alg.	
rep.	$\mathcal{P}(n)$
dim $\mathcal{P}(n)$	
Gen. series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	binary, cubic, set-theoretic.
Alternative	
Relationship	Strongly related with the notions of 2-category and 2-group
Unit	if a unit for both, then $* = \cdot$ and they are commutative
Comment	Many variations depending on the hypotheses on $*$ and \cdot Most common \cdot and $*$ are associative.
Ref.	

Name	Hypercommutative algebra
	underlying objects: graded vector spaces
Notation	<i>HyperCom</i>
Def. oper.	(x_1, \dots, x_n) n -ary operation of degree $2(n-2)$ for $n \geq 2$
sym.	totally symmetric
rel.	$\sum_{S_1 \sqcup S_2 = \{1, \dots, n\}} ((a, b, x_{S_1}), c, x_{S_2}) =$ $\sum_{S_1 \sqcup S_2 = \{1, \dots, n\}} (-1)^{ c x_{S_1} } (a, (b, x_{S_1}, c), x_{S_2}) ,$ for any $n \geq 0$.
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t) =$
Dual operad	Gravity algebra, see ref below
Chain-cplx	
Properties	
Alternative	
Relationship	
Unit	
Comment	
Ref.	[Get95]

Name	Associative algebra up to homotopy
	operad with underlying space in dgVect
Notation	A_∞
Def. oper.	m_n for $n \geq 2$ (operation of arity n and degree $n - 2$)
sym.	
rel.	$\partial(m_n) = \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{p+qr} m_k \circ (\text{id}^{\otimes p} \otimes m_q \otimes \text{id}^{\otimes r}).$
Free alg.	
\mathcal{P}_n	$(A_\infty)_n = \mathbb{K}[PT_n]$ isomorphic to $C_\bullet(\mathcal{K}^{n-2})$ as chain complex
$\dim \mathcal{P}_n$	where \mathcal{K}^n is the Stasheff polytope of dimension n
Gen. series	$f(t)$
Dual operad	
Chain-cplx	
Properties	multi-ary, quadratic, ns, minimal model for As
Alternative	Cobar construction on $As^!$: $A_\infty = As_\infty := \Omega As^!$
Relationship	Many, see the literature.
Unit	Good question !
Comment	There are two levels of morphisms between A_∞ -algebras: the morphisms and the ∞ -morphisms, see [LV11] for instance.
Ref.	[Sta63] J. Stasheff, <i>Homotopy associativity of H-spaces. I, II.</i> TAMS 108 (1963), 275-292 ; ibid. 108 (1963), 293312.

Name	Commutative algebra up to homotopy
	operad with underlying space in dgVect
Notation	C_∞
Def. oper.	m_n for $n \geq 2$ (operation of arity n and degree $n - 2$) which vanishes on the sum of $(p, n - p)$ -shuffles, $1 \leq p \leq n - 1$.
sym.	
rel.	$\partial(m_n) = \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{p+qr} m_k \circ (\text{id}^{\otimes p} \otimes m_q \otimes \text{id}^{\otimes r}).$
Free alg.	
	$\mathcal{P}(n)$
	$\dim \mathcal{P}(n)$
Gen. series	$f(t)$
Dual operad	
Chain-cplx	
Properties	multi-ary, quadratic, minimal model for Com
Alternative	Cobar construction on Com^i : $Com_\infty = Com_\infty := \Omega Com^i$
Relationship	
Unit	
Comment	
Ref.	[Kad85] T. Kadeishvili, <i>The category of differential coalgebras and the category of $A(\infty)$-algebras.</i> Proc. Tbilisi Math.Inst. 77 (1985), 50-70.

Name	Lie algebra up to homotopy
	operad with underlying space in dgVect
Notation	L_∞
Def. oper.	ℓ_n , n -ary operation of degree $n - 2$, for all $n \geq 2$
sym.	
rel.	$\sum_{\substack{p+q=n+1 \\ p,q>1}} \sum_{\sigma \in Sh_{p,q}^{-1}} \text{sgn}(\sigma)(-1)^{(p-1)q} (\ell_p \circ_1 \ell_q)^\sigma = \partial_A(\ell_n)$,
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t)$
Dual operad	
Chain-cplx	
Properties	multi-ary, quadratic, minimal model for <i>Lie</i>
Alternative	Cobar construction on <i>Lie</i> ⁱ : $Lie_\infty = Lie_\infty := \Omega Lie^i$
Relationship	
Unit	
Comment	
Ref.	

Name	Dendriform algebra up to homotopy
	operad with underlying space in dgVect
Notation	$Dend_{\infty}$
Def. oper.	$m_{n,i}$ is an n -ary operation, $1 \leq i \leq n$, for all $n \geq 2$
sym.	none
rel.	$\partial(m_{n,i}) = \sum (-1)^{p+qr} m_{p+1+r,\ell} (\underbrace{\text{id}, \dots, \text{id}}_p, m_{q,j}, \underbrace{\text{id}, \dots, \text{id}}_r)$ <p>sum extended to all the quintuples p, q, r, ℓ, j satisfying: $p \geq 0, q \geq 2, r \geq 0, p + q + r = n, 1 \leq \ell \leq p + 1 + q, 1 \leq j \leq q$ and either one of the following: $i = q + \ell$, when $1 \leq p + 1 \leq \ell - 1$, $i = \ell - 1 + j$, when $p + 1 = \ell$, $i = \ell$, when $\ell + 1 \leq p + 1$.</p>
Free alg.	
rep. \mathcal{P}_n	
$\dim \mathcal{P}_n$	
Gen. series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	multi-ary, quadratic, ns, minimal model for $Dend$
Alternative	
Relationship	
Unit	
Comment	
Ref.	See for instance [LV11]

Name	\mathcal{P} -algebra up to homotopy
	operad with underlying space in <code>dgVect</code>
Notation	\mathcal{P}_∞
	the operad \mathcal{P} is supposed to be quadratic and Koszul
Def. oper.	
sym.	
rel.	
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen. series	$f(t)$
Dual operad	
Chain-cplx	
Properties	quadratic, ns if $\mathcal{P}(n) = \mathcal{P}_n \otimes \mathbb{K}[\mathbb{S}_n]$.
Alternative	Cobar construction on \mathcal{P}^i : $\mathcal{P}_\infty := \Omega \mathcal{P}^i$
Relationship	
Unit	
Comment	
Ref.	See for instance [LV11]

Name	Brace algebra
Notation	<i>Brace</i>
Def. oper.	$\{x_0; x_1, \dots, x_n\}$ for $n \geq 0$
sym.	$\{x; \emptyset\} = x$
rel.	$\{\{x; y_1, \dots, y_n\}; z_1, \dots, z_m\} =$ $\sum \{x; \dots, \{y_1; \dots\}, \dots, \{y_n; \dots\}, \dots\}.$ the dots are filled with the variables z_i 's (in order).
Free alg.	
rep. $\mathcal{P}(n)$	$Brace(n) = \mathbb{K}[PBT_{n+1}] \otimes \mathbb{K}[\mathbb{S}_n]$
dim $\mathcal{P}(n)$	$1, 1 \times 2!, 2 \times 3!, 5 \times 4!, 14 \times 5!, 42 \times 6!, 132 \times 7!, \dots, c_{n-1} \times n!, \dots$
Gen. series	$f(t)$
Dual operad	
Chain-cplx	
Properties	multi-ary, quadratic, quasi-regular.
Alternative	
Relationship	$Brace\text{-}alg \rightarrow MB\text{-}alg$ $Brace\text{-}alg \rightarrow PreLie\text{-}alg$, ($\{-; -\}$ is a pre-Lie product) If A is a brace algebra, then $T^c(A)$ is a cofree Hopf algebra
Unit	
Comment	There exists a notion of brace algebra with differentials useful in algebraic topology
Ref.	[Ron02] Ronco, M. <i>Eulerian idempotents and Milnor-Moore theorem for certain non-cocommutative Hopf algebras.</i> J. Algebra 254 (2002), no. 1, 152–172.

Name	Multi-brace algebra
Notation	MB
Def. oper. sym. rel.	$(x_1, \dots, x_p; y_1, \dots, y_q)$ for $p \geq 1, q \geq 1$ \mathcal{R}_{ijk} , see ref
Free alg.	
rep. $\mathcal{P}(n)$	$MB(n) = (\mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n]) \otimes \mathbb{K}[\mathbb{S}_n], n \geq 2$
dim $\mathcal{P}(n)$	$1, 1 \times 2!, 6 \times 3!, 22 \times 4!, 90 \times 5!, ?? \times 6!, ??? \times 7!, \dots, 2C_n \times n!, \dots$ $C_n =$ Schröder number (super Catalan)
Gen. series	$f(t)$
Dual operad	
Chain-cplx	
Properties	multi-ary, quadratic, quasi-regular.
Alternative	Used to be denoted by B_∞ or \mathbf{B}_∞ confusing notation with respect to algebras up to homotopy
Relationship	$Brace\text{-}alg \rightarrow MB\text{-}alg$ $Brace\text{-}alg \rightarrow PreLie\text{-}alg$, ($\{-; -\}$ is a pre-Lie product) If A is a brace algebra, then $T^c(A)$ is a cofree Hopf algebra (and vice-versa)
Unit	
Comment	There exists a notion of MB-infinity algebra with differentials useful in algebraic topology (and called B_∞ -algebra)
Ref.	[LR06] Loday, J.-L., and Ronco, M. <i>On the structure of cofree Hopf algebras</i> J. reine angew. Math. 592 (2006) 123–155.

Name **Double Poisson algebra**

Notation $2Pois$

Def. oper.
 sym.
 rel.

Free alg.

rep. $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$ $1, \dots, ??, \dots$

Gen. series $f(t)$

Dual operad

Chain-cplx

Properties

Alternative

Relationship

Unit

Comment

Ref. M. Van den Bergh,

Name	\mathcal{X}^\pm -algebra
Notation	\mathcal{X}^\pm -alg
Def. oper.	$x \nwarrow y, x \nearrow y, x \searrow y, x \swarrow y$.
rel.	
	$ \begin{aligned} (\nwarrow) \nwarrow &= \nwarrow (\nwarrow) + \nwarrow (\swarrow), & (\swarrow) \nwarrow &= \swarrow (\nwarrow), & (\nwarrow) \swarrow + (\swarrow) \swarrow &= \swarrow (\swarrow), \\ (\nwarrow) \nwarrow &= \nwarrow (\searrow) + \nwarrow (\nearrow), & (\swarrow) \nwarrow &= \swarrow (\nearrow), & (\nwarrow) \swarrow + (\swarrow) \swarrow &= \swarrow (\searrow), \\ (\nearrow) \nwarrow &= \nearrow (\nwarrow) + \nearrow (\swarrow), & (\searrow) \nwarrow &= \searrow (\nwarrow), & (\nearrow) \swarrow + (\searrow) \swarrow &= \searrow (\swarrow), \\ (\nwarrow) \nearrow &= \nearrow (\nearrow) + \nearrow (\searrow), & (\swarrow) \nearrow &= \searrow (\nearrow), & (\nwarrow) \searrow + (\swarrow) \searrow &= \searrow (\searrow), \\ (\nearrow) \nearrow &= \nearrow (\nearrow) + \nearrow (\searrow), & (\searrow) \nearrow &= \searrow (\nearrow), & (\nearrow) \searrow + (\searrow) \searrow &= \searrow (\searrow). \end{aligned} $
	$ \begin{aligned} (\nearrow) \searrow - (\nwarrow) \searrow &= + \nwarrow (\swarrow) - \nwarrow (\searrow), & (16+) \\ (\nearrow) \searrow - (\nwarrow) \searrow &= - \nwarrow (\swarrow) + \nwarrow (\searrow). & (16-) \end{aligned} $
Free alg.	
rep. \mathcal{P}_n	
$\dim \mathcal{P}_n$	
Gen. series	$f(t) =$
Dual operad	Both self-dual.
Chain-cplx	
Properties	binary, quadratic, ns.
Alternative	
Relationship	A quotient of $Dend \blacksquare Dias$ (Manin black product in ns operads) Fit into the “operadic butterfly” diagram, see the reference.
Unit	
Comment	
Ref.	[Lod06] Loday J.-L., <i>Completing the operadic butterfly</i> , Georgian Math Journal 13 (2006), no 4. 741–749.

Name *put your own type of* **algebras**

Notation

Def. oper.
 sym.
 rel.

Free alg.

rep. $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$

Gen. series $f(t) =$

Dual operad

Chain-cplx

Properties

Alternative

Relationship

Unit

Comment

Ref.

Integer sequences which appear in this paper, up to some shift and up to multiplication by $n!$ or $(n-1)!$.

1	1	1	1	1	...	1	...	<i>Com, As</i>
1	2	0	0	0	...	0	...	<i>Nil</i>
1	2	2	2	2	...	2	...	<i>Dual2as</i>
1	2	3	4	5	...	n	...	<i>Dias, Perm</i>
1	2	5	12	15	...	??	...	<i>Altern[!]</i>
1	2	5	14	42	...	c_{n-1}	...	<i>Mag, Dend, brace, Dup</i>
1	2	6	18	57	...	f_{n+2}	...	<i>MagFine</i>
1	2	6	22	90	...	$2C_n$...	<i>Dipt, 2as, brace</i>
1	2	6	24	120	...	$n!$...	<i>As, Lie, Leib, Zinb</i>
1	2	7			...	??	...	<i>Lie-adm</i>
1	2	7	32	175	...	??	...	<i>Altern</i>
1	2	7	40		...	??	...	<i>Moufang</i>
1	2	9	64	625	...	n^{n-1}	...	<i>PreLie, NAP</i>
1	2	10	26	76	...	??	...	<i>Parastat</i>
1	3	7	15	31	...	$(2^n - 1)$...	<i>Trias</i>
1	3	9			...	??	...	<i>Malcev</i>
1	3	11	45	197	...	C_n	...	<i>TriDend</i>
1	3	13	75	541	...	??	...	<i>CTD</i>
1	3	16	125	6^5	...	$(n+1)^{n-1}$...	<i>Park</i>
1	3	20	210	3024	...	$a(n)$...	<i>PostLie</i>
1	3	50			...	??	...	<i>Jordan triples</i>
1	4	9	16	25	...	$n^2 n!$...	<i>Quadri[!]</i>
1	4	23	156	1162	...	??	...	<i>Quadri</i>
1	4	23	181		...	??	...	see <i>PreLie</i>
1	4	27	256		...	n^n	...	
1	8				...	??	...	<i>Akivis</i>
1	8	78	1104		...	??	...	<i>Sabinin</i>

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